TURBULENCE WITH LOW HETEROGENEITY AND ANISOTROPY

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It is shown that using expansions with respect to the average velocity gradient, retaining only the first two terms of the expansion, one can satisfactorily describe the turbulence in a pipe in the region of the core of the stream.

Real large-scale turbulence does not fit into the framework of a simple mathematical description using uniform isotropic random fields. Nevertheless, in many types of flow there are regions with a small deviation of the flow from uniform and isotropic and here a simplified description is possible using one or another expansion with respect to the degree of departure from the isotropic state.

If the scales of the heterogeneities of the turbulent pulsation fields are much smaller than the scales of the heterogeneity of the average field, then all the heterogeneity and anisotropy of the pulsations in a first approximation are due to the orienting effect of the average field. Then the average characteristics of the pulsation fields depend isotropically on the characteristics of the main stream and one can be limited to the first terms of expansions with respect to these values.

A concrete example of established turbulent flow in a round pipe far from its walls is analyzed later. Strictly speaking, the assumption that the scales of the average and pulsation fields differ strongly is not correct here (on the whole, it is satisfied extremely rarely for real turbulent flows).* Therefore, a nonlocal description of the dependence of the pulsations on the average field would be more suitable. Nevertheless, even a simplified local description of the controlling effect of the anisotropy of the average stream on the pulsations proves to be satisfactory here, as will be seen.

The flow in a round pipe has axial symmetry, and all the characteristics of the turbulence, generally speaking, can depend on the unit vector of the direction of the average stream $\vec{\lambda} = \langle \vec{u} \rangle / |\langle \vec{u} \rangle|$ [1]. If one assumes that the dependence of the pulsations on the average stream is local and they depend on the lower derivatives of the average velocity in an invariant way relative to arbitrary axisymmetric motions (and reflections), then the single-point tensor characteristics of the pulsation field must be isotropic tensor functions of the vector $\vec{\lambda}$ and of the average deformation velocity tensor $\langle e_{ij} \rangle = 1/2 \langle \partial u_i/\partial x_j + \partial u_j/\partial x_i \rangle$.

For the values $R_{ij}(\vec{x}, \vec{x}) \equiv \langle u_i'(\vec{x}) u_j'(\vec{x}) \rangle$ and $P_i \equiv \langle p'(\vec{x}) u_i'(\vec{x}) \rangle$ such isotropic tensor functions for the developed flow far from the wall of the pipe have the form

$$R_{ij} = a_0 u_*^2 \delta_{ij} + 2a_1 u_* r_0 \langle e_{ij} \rangle + 2a_2 r_0^2 \langle e_{i\alpha} \rangle \langle e_{\alpha j} \rangle + \\ + a_3 u_*^2 \lambda_i \lambda_j + 2a_4 u_* r_0 (\langle e_{i\alpha} \rangle \lambda_\alpha \lambda_j + \langle e_{j\alpha} \rangle \lambda_\alpha \lambda_i) + \\ + a_5 r_0^2 (\langle e_{i\beta} \rangle \langle e_{\beta\alpha} \rangle \lambda_\alpha \lambda_j + \langle e_{j\beta} \rangle \langle e_{\beta\alpha} \rangle \lambda_\alpha \lambda_i), \tag{1}$$

$$P_i = (b_0 u_*^3 \delta_{i\alpha} + 2b_1 u_*^2 r_0 \langle e_{i\alpha} \rangle + 2b_2 u_* r_0^2 \langle e_{i\beta} \rangle \langle e_{\beta\alpha} \rangle) \lambda_\alpha, \tag{2}$$

with the dimensionless coefficients a_k and b_k depending on the dimensionless invariants $I_0 \equiv \lambda_\alpha^2 = 1$, and $I_1 \equiv r_0 \langle e_{\alpha\alpha} \rangle / u_* = 0$:

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* It is well known that such an assumption is violated, first of all, for pulsations in the direction of the main stream. It is just for the longitudinal pulsations, as will be seen, that the simplest description proves to be the least accurate (see Fig. 1).

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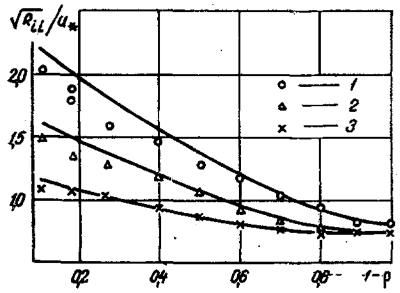

Fig. 1

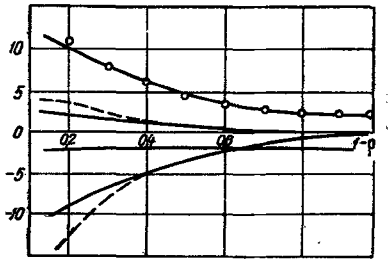

Fig. 2

Fig. 1. Distributions of intensity of turbulent pulsations in the core of a stream in a pipe. The points correspond to Laufer's data [2]: 1) $\sqrt{\langle u_x'^2\rangle}/u_*$; 2) $\sqrt{\langle u_\varphi'^2\rangle}/u_*$; 3) $\sqrt{\langle u_r'^2\rangle}/u_*$; the curves correspond to Eqs. (8).

Fig. 2. Pulsation energy balance in the core of a stream in a pipe. The solid curves correspond to Eqs. (10) while the points and dashed lines correspond to Laufer's data [2].

$$I_2 \equiv (r_0 \langle e_{\alpha\beta}\rangle/u_*)^2,$$
$$I_3 \equiv r_0^3 \langle e_{\alpha\beta}\rangle\langle e_{\beta\gamma}\rangle\langle e_{\gamma\alpha}\rangle/u_*^3,$$
$$I_4 \equiv r_0 \langle e_{\alpha\beta}\rangle \lambda_\alpha\lambda_\beta/u_*, \quad I_5 \equiv r_0^2 \langle e_{\alpha\beta}\rangle\langle e_{\beta\gamma}\rangle \lambda_\alpha\lambda_\gamma/u_*^2,$$

since for the flow of an incompressible* fluid near the axis of a pipe the characteristic scale values are the dynamic velocity u_* and the pipe radius r_0.

The dissipative tensor $D_{ij} \equiv \nu\langle(\partial u_i'/\partial x_\alpha)(\partial u_j'/\partial x_\alpha)\rangle$ and the stream vector of the kinetic energy of the pulsations $Q_i(\vec{x}, \vec{x}) \equiv \langle u_i'(\vec{x})u_\alpha'^2(\vec{x})/2$ have an analogous general form.

In a cylindrical coordinate system whose axis coincides with the pipe axis the average deformation velocity tensor has only two nonzero components $\langle e_{rx}\rangle = \langle e_{xr}\rangle = 1/2\langle\partial u/\partial r\rangle$, while the vector $\vec{\lambda}$ has the one component $\lambda_x = 1$.

Since only $I_0 = 1$, $I_2 = I_5 = I \equiv 1/2\langle\partial u^+/\partial\rho\rangle^2$ among the invariants are different from zero, in the indicated coordinate system it follows from (1) and (2) that

$$\begin{gathered}R_{xx}/u_*^2 = a_0 + a_3 + (a_2 + a_5)I, \\ R_{\varphi\varphi}/u_*^2 = a_0, \quad R_{rr}/u_*^2 = a_0 + a_2 I, \\ R_{xr}/u_*^2 = (a_1 + a_4)\langle\partial u^+/\partial\rho\rangle, \quad R_{x\varphi} = R_{r\varphi} = 0, \\ P_x/u_*^3 = b_0 + b_2 I, \quad P_r/u_*^3 = b_1\langle\partial u^+/\partial\rho\rangle, \quad P_\varphi = 0.\end{gathered} \tag{3}$$

Here the coefficients a_k and b_k depend on I, while $\rho \equiv r/r_0$. Analogous equations can be written for D_{ij} and Q_i.

Near the pipe axis the value of the average velocity gradient $(2I)^{1/2} = \langle\partial u^+/\partial\rho\rangle$ is small. Let us assume that the functions $a_k(I)$ and $b_k(I)$ are expanded in series with respect to whole powers of I and let us keep only the first two terms in these expansions. Then for the Reynolds shear stress we obtain (a_{km} are numerical coefficients)

$$R_{xr}/u_*^2 \approx (a_{10} + a_{40})\left\langle\frac{\partial u^+}{\partial\rho}\right\rangle + \frac{1}{2}(a_{11} + a_{41})\left\langle\frac{\partial u^+}{\partial\rho}\right\rangle^3 + \ldots \tag{4}$$

On the other hand, it is known [2, 3] that for regions far from the wall it follows from the equation of conservation of momentum that

$$R_{xr}/u_*^2 = \rho. \tag{5}$$

*An incompressible medium with a density different from unity is analyzed later.

Hence it is seen that in the first approximation the average velocity profile near the pipe axis must be parabolic.

Such a parabolic approximation of the average velocity profile for the flow in pipes is well known [2, 3]. According to Fig. 7.49 in [2], the experimental data of Laufer are satisfactorily described by the equation

$$U^+ - \langle u^+ \rangle \approx 7.2\,\rho^2 \tag{6}$$

up to $\rho \approx 0.9$ (a different value of the coefficient is given in the book [3]: 7.6).

Let us compare the expansions for other values with Laufer's data, which are presented in [2]. From (3) we get the equations

$$\begin{aligned} R_{xx}/u_*^2 &\approx a_{00} + a_{30} + (a_{01} + a_{31} + a_{20} + a_{50}) I + \ldots, \\ R_{\varphi\varphi}/u_*^2 &\approx a_{00} + a_{01} I + \ldots, \\ R_{rr}/u_*^2 &\approx a_{00} + (a_{01} + a_{20}) I + \ldots, \\ P_r/u_*^3 &\approx (b_{10} + b_{11} I)\langle \partial u^+/\partial\rho \rangle + \ldots \end{aligned} \tag{7}$$

which represent expansions with respect to ρ^2, since from (4)-(6) we have $I = I(\rho^2) \approx (10\rho)^2 + \ldots$

The experimental data for the root-mean-square velocity pulsations are satisfactorily described by the quadratic expressions

$$R_{xx}/u_*^2 \approx 0.65 + 5\rho^2, \quad R_{\varphi\varphi}/u_*^2 \approx 0.55 + 2.5\rho^2, \quad R_{rr}/u_*^2 \approx 0.55 + \rho^2 \tag{8}$$

with $\rho \leqslant 0.4$, 0.6, and 0.8 for R_{xx}, $R_{\varphi\varphi}$, and R_{rr}, respectively, as is seen from Fig. 1. The numerical values of the coefficients of (8) are obtained with the substitution of $a_{00} \approx 0.55$, $a_{30} \approx 0.1$, $a_{01} \approx a_{31} + a_{50} \approx 0.04$, and $a_{20} \approx -0.015$. It should be noted that in the region of $\rho \leqslant 0.9$ the departure of the points from the curves described by Eqs. (8) is small. This indicates the relative smallness of the coefficients of the succeeding terms of the expansion with respect to ρ^2.

Experimental estimates of the variation in the different terms of the equation of pulsation-energy balance in the core of a turbulent stream in a pipe

$$r_0 D_{\alpha\alpha}/u_*^3 + \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho P_r/u_*^3) = -\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho Q_r/u_*^3) - R_{xr}/u_*^2 \left\langle \frac{\partial u^+}{\partial\rho} \right\rangle \tag{9}$$

are presented in [2] in Figs. 7.42 and 7.44 and in [3] in Fig. 9.12.

Using the expansions which have been discussed, it is not hard to verify that the equations

$$\begin{aligned} r_0 D_{\alpha\alpha}/u_*^3 &\approx 2.1 + 12\rho^2, \quad P_r/u_*^3 \approx 0.7\rho^3, \\ -R_{xr}/u_*^2 \langle \partial u^+/\partial\rho \rangle &\approx 14.4\rho^2, \quad -Q_r/u_*^3 \approx 0.9\rho + 0.2\rho^3 \end{aligned} \tag{10}$$

satisfactorily describe the behavior of the different terms* of Eq. (9) when $\rho \leqslant 0.7$. This is well seen from Fig. 2, in which Laufer's data for the dissipation when $2r_0U/\nu = 5 \cdot 10^4$ are shown by points while the curves of Fig. 7.44 from [2] are shown by dashes where they deviate from the solid curves corresponding to the approximations (10).

Thus, the experimental data for the core of a turbulent stream in a pipe can be described successfully with the help of the first two terms of an expansion with respect to the average velocity gradient with a higher accuracy than could be expected on the basis of general considerations.

In conclusion, we note that neither the viscosity nor the size of the pipe plays an important role for developed turbulent flow of a viscous fluid in the region of $0.1 \geqslant 1 - \rho > 100/r_0^+$, and therefore, the functions here are different from (6), (8), and (10). In particular, the average velocity profile has the logarithmic form [2, 3]

$$U^+ - \langle u^+ \rangle \approx -2.5 \ln(1 - \rho) + B_1. \tag{11}$$

* The slight incompatibility of the numerical coefficients in (10), revealed when these expressions are substituted into (9), is covered by the inaccuracy of the figures in [2, 3] and evidently does not exceed the inaccuracy of the measurements.

If one approximates the entire velocity profile when $1 - \rho > 100/r_0^+$ by the two simplified expressions (6) and (11), which pass smoothly into one another, then for B_1 and for the position of the matchup point $\eta^0 = 1 - \rho^0$ we obtain the estimates

$$\eta^0 \approx 0.22, \; B_1 \approx 0.6. \qquad (12)$$

NOTATION

$\langle \vec{u} \rangle$, average velocity vector; $\vec{\lambda}$, unit vector of stream direction; e_{ij}, deformation velocity tensor; R_{ij}, Reynolds stress tensor; $D_{\alpha\alpha}$, pulsation energy dissipation; Q_i, P_i, stream vectors of pulsation energy and pressure; r_0, pipe radius; r, distance from axis; $\rho = r/r_0$, dimensionless distance from axis; u_*, dynamic velocity; I_k, simultaneous invariants of vector λ_i and tensor $\langle e_{ij} \rangle$; a_k, b_k, invariant functions; a_{kn}, b_{kn}, numerical coefficients; $u^+ = u/u_*$, dimensionless velocity; η^0, boundary of turbulent core.

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